3^{rd} **Annual Mathematics Olympiad**

Group Competition

Barry University

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School Name: ____

Given a natural number $n_0 > 1$, players A and B are choosing natural numbers n_1, n_2 ... in order by the following rule. Player A, knowing the number n_{2k}, can choose any number n_{2k+1} so that

$$n_{2k} \leq n_{2k+1} \leq n_{2k}^2.$$

Then, player B chooses any number n_{2k+2} so that $\frac{n_{2k+1}}{n_{2k+2}}$ is a positive integer power of a prime number. Player A is a winner if he/she chooses number 1990, and player B is a winner if he/she chooses number 1. Find all numbers n_0 for which

a) Player A has a winning strategy;

b) Player B has a winning strategy;

c) Neither player A nor player B have a winning strategy.

Please justify your answer!

Solution:

a) $n_0 \geq 8$; b) $2 \leq n_0 \leq 5$; c) $n_0 = 6$; $n_0 = 7$.

Let W be a set of all natural numbers n_0 for which player A has a winning strategy. We are going to prove the following:

Lemma. Let $\{m, m+1, ..., 1990\} \subset W$, $s \leq 1990$ and $\frac{s}{p^r} \geq m$, where p^r is the largest power of p which divides s. Then all natural numbers n_0 such that

$$\sqrt{s} \leq n_0 < m$$

also belong to W.
Proof. If $\sqrt{s} \leq n_0 < m$, player A can choose $n_1 = s$. Then player B, must choose one of the numbers n_2, such that

$$m \leq \frac{s}{p^r} \leq n_2 < s \leq 1990.$$

Hence, $n_2 \in W$ and player A will win. □

Since $45^2 = 2025 > 1990$, then all n_0 for which $45 \leq n_0 \leq 1990$, obviously belong to W. Numbers $m = 45$ and $s = 420 = 2^2 \times 3 \times 5 \times 7$ satisfy hypothesis of the lemma, since $\sqrt{420} < 21 < 45$; hence, $\{21, 22, ..., 44\} \subset W$. Using the lemma again for $m = 21$ and $s = 168 = 2^3 \times 3 \times 7$, we get that $\{13, 14, ..., 20\} \subset W$. For $m = 13$ and $s = 105 = 3 \times 5 \times 7$ we get $\{11, 12\} \subset W$. Similarly, for $m = 11$ and $s = 60 = 2^2 \times 3 \times 5$ we get $\{8, 9, 10\} \subset W$.

Therefore, we have

$$\{8, 9, ..., 1990\} \subset W.$$

If $n_0 > 1990$, player A can choose natural number r such that

$$2^r \times 3^2 < n_0 \leq 2^{r+1} \times 3^2 < n_0^2,$$

and therefore choose $n_1 = 2^{r+1} \times 3^2$. Now, it does not matter what number would player B choose, it is important that $8 \leq n_2 < n_0$. After few moves we get that $8 \leq n_{2k} \leq 1990$. It follows that for all $n_0 \geq 8$, player A has a winning strategy.

Let $n_0 \leq 5$. Since the smallest product of three prime numbers is $2 \times 3 \times 5 = 30 > 5^2$, player A can choose a number $n_1 = \frac{p^r}{q^s}$, where p is a prime and q is either prime or number 1, $p^r > q^s$ and r and s are natural numbers. Then player B can choose

$$n_2 = q^s = \frac{n_1}{p^r} < \sqrt{n_1} \leq n_0.$$

After final number of moves, player B gets $n_{2k} = 1$ and wins.

For $n_0 = 6$ or $n_0 = 7$ player A can choose $n_1 = 30 = 2 \times 3 \times 5$ or $n_1 = 2 \times 3 \times 7$ and then B chooses $n_2 = 6$. After that players A and B have to choose alternatively numbers $30, 6, 30, 6, ...$; otherwise the opponent will get to the winning strategy.