3^{rd} Annual Mathematics Olympiad
Barry University

Individual Competition

November 30, 2017

Name: ________________________________

School: ________________________________

<u>Justify all answers!</u>

1. Solve $x = \sqrt{2 + \sqrt{2 - \sqrt{2 + x}}}$ in Real number system.
2. Let x, y, and z be Natural numbers for which $x^3 - y^3 - z^3 = 3xyz$ and $x^2 = 2(y + z)$. Find $x + y + z$.
3. On the bisector of $\angle BAC$ of a triangle ABC there are two points B_1 and C_1 such that $BB_1 \perp AB$, $CC_1 \perp AC$. Let M be a midpoint of $\overline{B_1C_1}$. Prove that $\overline{MB} = \overline{MC}$.
4. From all the triangles with the same perimeter, equilateral triangle has the largest area. Prove it.
5. Let $n > 2$ be a Natural number and let positive real numbers $a_0, a_1, ...$ satisfy

$$(a_{k-1} + a_k)(a_k + a_{k+1}) = a_{k-1} - a_{k+1}$$

for all $k = 1, 2, ..., n - 1$. Prove that $a_n < \frac{1}{n-1}$.

Solutions:

1. Equality is defined for $x \in [-2, 2]$, because $x + 2 \geq 0$ and $\sqrt{x+2} \leq 2$. Let $x = 2\cos t$ for $t \in [0, \pi]$. Then $\cos\frac{t}{2} \geq 0$ and

$$\sqrt{2+x} = \sqrt{2(1+\cos t)} = \sqrt{4\cos^2\frac{t}{2}} = 2\cos\frac{t}{2}$$

Further,

$$\sqrt{2 - 2\cos\frac{t}{2}} = \sqrt{2\left(1 - \cos\frac{t}{2}\right)} = \sqrt{4\sin^2\frac{t}{4}} = 2\sin\frac{t}{4}$$

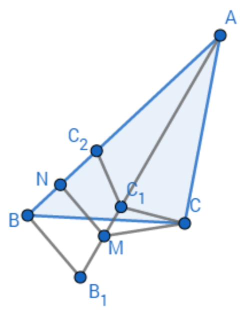
because $\sin\frac{t}{4} \geq 0$. Moreover,

$$\sqrt{2 + 2\sin\frac{t}{4}} = \sqrt{2\left(1 + \cos\left(\frac{\pi}{2} - \frac{t}{4}\right)\right)} = \sqrt{4\cos^2\left(\frac{\pi}{4} - \frac{t}{8}\right)} = 2\cos\left(\frac{\pi}{4} - \frac{t}{8}\right)$$

because $\cos\left(\frac{\pi}{4} - \frac{t}{8}\right) \geq 0$. Finally, equality becomes $\cos t = \cos\left(\frac{\pi}{4} - \frac{t}{8}\right)$, $t \in [0, \pi]$, i.e., $t = 2\pi/9$ or $x = 2\cos 2\pi/9$. **(20 points)**

2. Since $3xyz > 0$, it follows that $x > y, z$, and $4x > 2(y+z) = x^2$, i.e., $x < 4$ **(5 points)**. From $x^2 = 2(y+z)$ it follows that $2|x$, so we get $x = 2$ **(15 points)**. Hence $y = z = 1$, and $x + y + z = 4$. **(20 points)**

3. Let C_2 and N be points on $\overline{AB}$ such that $C_1C_2 \perp AB$ and $MN \parallel B_1B$. Since the triangles AC_1C and AC_1C_2 are congruent, it follows that $C_1C = C_1C_2$ **(5 points)**. Since M is midpoint of B_1C_1 and $C_1C_2 \perp AB$, it follows that N is midpoint of BC_2. Therefore, the height MN of triangle BMC_2 is a median, and the triangle is isosceles, i.e., $BM = MC_2$ **(15 points)**. On the other hand, since the triangles MC_1C_2 and MC_1C are congruent, it follows that $MC = MC_2$. Therefore, $BM = MC_2 = MC$. **(20 points)**



4. Area of equilateral triangle is $P_1 = \frac{a^2\sqrt{3}}{4}$. Set $a = \frac{2s}{3}$, where s is half of perimeter value. Then $P_1 = \left(\frac{2s}{3}\right)^2 \frac{\sqrt{3}}{4}$. Area of an arbitrary triangle we can calculate by Heron's formula so

inequality that we have to prove becomes

$$P = \sqrt{s(s-a)(s-b)(s-c)} \leq \left(\frac{2s}{3}\right)^2 \frac{\sqrt{3}}{4} \quad (\mathbf{5points}).$$

Squaring above inequality and simplifying we get

$$(s-a)(s-b)(s-c) \leq \left(\frac{s}{3}\right)^3$$

To show this, observe that inequality between arithmetic and geometric means is

$$(s-a)(s-b)(s-c) \leq \left(\frac{s-a+s-b+s-c}{3}\right)^3 \quad (\mathbf{10points})$$
$$(s-a)(s-b)(s-c) \leq \left(\frac{3s-2s}{3}\right)^3$$
$$(s-a)(s-b)(s-c) \leq \left(\frac{s}{3}\right)^3 \quad (\mathbf{15points})$$

Equality holds when $s-a = s-b = s-c$, i.e., when $a = b = c$. **(20 points)**

5. Given equality is equivalent to

$$\frac{1}{a_{k+1} + a_k} = 1 + \frac{1}{a_k + a_{k-1}} \quad (\mathbf{8points})$$

for all k. By induction we get

$$\frac{1}{a_{k+1} + a_k} = k + \frac{1}{a_0 + a_1}. \quad (\mathbf{13points})$$

Setting $k = n-1$ above, we get

$$\frac{1}{a_n + a_{n-1}} = n - 1 + \frac{1}{a_0 + a_1}.$$

Since $\frac{1}{a_n} > \frac{1}{a_n + a_{n-1}}$, and $\frac{1}{a_0+a_1} > n-1$ **(18 points)** we get $a_n < \frac{1}{n-1}$. **(20 points)**