4^{th} Annual Mathematics Olympiad Barry University

Individual Competition

November 9, 2018

Name: ________________________________

School: ________________________________

Justify all answers!

1. If

$$A = 1 \cdot 2 + 3 \cdot 4 + 5 \cdot 6 + ... + 2017 \cdot 2018$$

and

$$B = 2 \cdot 3 + 4 \cdot 5 + 6 \cdot 7 + ... + 2018 \cdot 2019,$$

find

$$\frac{B - A}{1010}$$

2. In a triangle $\triangle ABC$ with sides $BC = a$, $CA = b$, and $AB = c$ a circle is inscribed. If one tangent of that circle crosses sides AC and BC at points P and Q respectively, find the perimeter of the triangle $\triangle PQC$.

3. A clock has three hands that rotate at a constant speed. Second hand rotates full circle in one minute, minute hand rotates full circle in one hour, and hour hand rotates full circle in 12 hours. At midnight, all hands are in the same position. How many times from that moment in a period of 24 hours will one hand make $30°$ angle with the other two.

4. Suppose that the pair of real numbers (x, y) satisfies the system

$$x^2 - 3xy + 2y^2 + x - y = 0$$
$$x^2 - 2xy + y^2 - 5x + 7y = 0.$$

Show that (x, y) also satisfies the equation

$$xy - 12x + 15y = 0$$

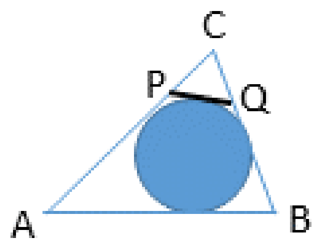
5. Find all Natural numbers a, b, and c that satisfy

$$a^3 + b^3 + c^3 = 2001.$$

Solutions:

1. $B - A = 2(2+4+6+\cdots+2018) = 4(1+2+3+\cdots+1009) = 4\cdot 1009\cdot 1010/2 = 2\cdot 1009\cdot 1010$. So, $B - A = 2018$.

2. The quadrilateral $ABQP$ is tangential, so $AB + PQ = AP + BQ$. This implies that $AB + PQ = AC - CP + BC - CQ$, hence $P_{\triangle PQC} = a + b - c$.



3. Let $t > 0$ be number of seconds that passed after midnight. Angles that second, minute and hour hand have passed are, respectfully, 6t, $t/10$ and $t/120$ (in degrees). Condition that seconds and minute hand form angle of 30° is $6t - t/10 = \pm 30 + 360a$ which is equivalent with

$$t = \frac{300(\pm 1 + 12a)}{59};$$

condition that seconds and hour hand form angle of 30° is $6t - t/120 = \pm 30 + 360b$ which is equivalent with

$$t = \frac{3600(\pm 1 + 12b)}{719};$$

condition that minute and hour hand form angle of 30° is $t/10 - t/120 = \pm 30 + 360c$ which is equivalent with

$$t = \frac{3600(\pm 1 + 12c)}{11}$$

where a, b, c are some real numbers. By the hypothesis of the problem, two of three equations have to hold. Since pairs of numbers 11, 59, and 719 are relatively prime, it follows that t must be a whole number (precisely, denominator of the number t must divide two of these three numbers so it must be 1). Since one of the last two equality must hold, and number 3600 is relatively prime with 11 and 719, it follows that t divides 3600. In another words, the hypothesis will be satisfied after whole number of hours. Hence, second and minute hand are at 12 and then hour hand must be at 1 or 11. In 24hr period the hypothesis of the problem will be satisfied four times, namely at 1am, 11am, 1pm, and 11pm.

4. The first equation in the system factors: $(x-y)(x-2y+1)=0$. This gives us two systems

$$\begin{aligned} &x = y \\ &x^2 - 2xy + y^2 - 5x + 7y = 0 \end{aligned} \tag{1}$$

$$\begin{aligned} &x - 2y + 1 = 0 \\ &x^2 - 2xy + y^2 - 5x + 7y = 0 \end{aligned} \tag{2}$$

The solution to (1) is by substitution, giving $2y = 0$, hence $x = y$, and $(x, y) = (0, 0)$.
The Solution to (2): from first equation we have $x = 2y - 1$; substituting in the second equation gives

$$(2y-1)^2 + 2(2y-1)y + y^2 - 5(2y-1) + 7y = 0,$$

which reduces to $y^2 - 5y + 6 = 0$, and therefore $y = 2 \implies x = 3$, $y = 3 \implies x = 5$. So Solutions to original system are $(x, y) = (0, 0), (3, 2), (5, 3)$. Direct verification shows that they satisfy the equation $xy - 12x + 15y = 0$.

5. Assume without loss of generality that $a \le b \le c$. Note that $x^3 \equiv 1 \pmod 9$ or $x^3 \equiv 0 \pmod 9$, or $x^3 \equiv 8 \pmod 9$, for every $x \in \{0, 1, 2, \cdots, 8\}$. Also, $2001 \equiv 3 \pmod 9$. Using the given, it follows that $a^3 + b^3 + c^3 \equiv 3 \pmod 9$, so it must be that $a^3 \equiv 1 \pmod 9$, $b^3 \equiv 1 \pmod 9$ and $c^3 \equiv 1 \pmod 9$. Since $13^3 > 2001$, we get $a \le b \le c \le 12$, so numbers a, b, and c belong to $\{1, 4, 7, 10\}$. After further investigation we get that the only solutions are $(a, b, c) \in \{(10, 10, 1), (10, 1, 10), (1, 10, 10)\}$.