5^{th} Annual Mathematics Olympiad

Individual Competition

Barry University

November 21, 2019

Name: ____

School Name: ______________________________________

For the questions 1-8) please choose one of the answers A-E).

1) Determine the value of the parameter p for which the two roots of the quadratic equation

$$(p-1)\,x^2 - 2(p+1)\,x + p - 2 = 0$$

are equal. Once the value of p has been determined, what should the solution be to the corresponding quadratic equation?

A) -3/2

B) 2/5

C) 4/5

D) -5/3

E) 3/5

2) Find maximum of $|8x^2 - 2x - 3|$ for all x in the interval [0,1].

A) 3.5

B) 3.25

C) 3.125

D) 3

E) None of the above

3) Let $n \geq 3$ be a natural number. Which of the following is true?

I. $\sqrt[n+1]{n+1} < \sqrt[n]{n}$

II. $(n+1)^n > n^{n+1}$

III. $(1+\frac{1}{n})^n < n$

A) II and III

B) I and II

C) III only

D) I and III

E) None of the above

4) Consider a rhombus ABCD. On each of its sides, construct (outwardly) a square. Let K, L, M, N be the centers of the four squares. Then the figure KLMN is a:

A) Parallelogram

B) Rhombus

C) Rectangle

D) Square

E) General quadrilateral

5) Given a circle $\mathcal{C}$ with center O, compute the probability that a randomly chosen inscribed triangle $\triangle ABC$ contains O.

A) 1/4

B) 1/3

C) 2/5

D) 1/5

E) 3/4

6) Find smallest and largest natural number x for which $a = \sqrt{66 - \sqrt{x+1}}$ is also a natural number.

A) $\{6, 4222\}$

B) $\{3, 4224\}$

C) $\{4, 4324\}$

D) $\{3, 4095\}$

E) $\{3, 4355\}$

7) When the last three digits of the six digit number M are moved to the front, in the same order, one gets a number six times bigger then the number M. Find M?

A) 142142

B) 111857

C) 135740

D) 124854

E) 142857

8) Consider the following sequence:

$$\begin{aligned} a_1 &= \frac{x^2 \cdot x^3}{x} \\ a_2 &= \frac{x^2 \cdot x^3}{x} \cdot \frac{x^3 \cdot x^4}{x^2} \\ a_3 &= \frac{x^2 \cdot x^3}{x} \cdot \frac{x^3 \cdot x^4}{x^2} \cdot \frac{x^4 \cdot x^5}{x^3} \\ &\vdots \\ a_k &= \frac{x^2 \cdot x^3}{x} \cdot \frac{x^3 \cdot x^4}{x^2} \cdot \frac{x^4 \cdot x^5}{x^3} \cdots \frac{x^{k+1} \cdot x^{k+2}}{x^k} \end{aligned}$$

For what value of k, should the term of the sequence a_k be equal to x^{2010}?

A) 65

B) 56

C) 62

D) 60

E) 58

Please justify your answer for numbers 9) and 10)!

9) Show that for any convex polygon $\mathcal{P}$, with vertices $V_1, \ldots, V_n$, of unit area, there exists a parallelogram of area 2 that contains the polygon.

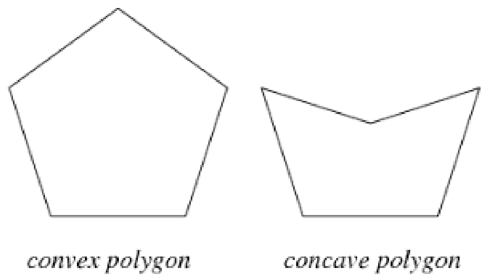


Figure 1: Example of a convex and non-convex polygon!

10) Show that sixteen added to the product of four consecutive odd numbers is a square of a natural number.

Solutions:

1) The two roots are equal when the discriminant is 0. This gives $p = 1/5$. Substituting that value in the equation and solving for x, gives $x = -3/2$. The correct answer is A).

2) $x = -b/(2a) = 1/8$, $f(1/8) = 31/8 = 3.125$, $f(0) = 3$, $f(1) = 3$. The correct answer is C).

3) It is easy to see that I $\iff$ opposite of II $\iff$ III. To show III we use mathematical induction. When $n = 3$, inequality $(1 + \frac{1}{3})^3 = 64/27 < 3$ is true. Next, assume inequality holds for arbitrary $n > 3$, and show it is true for $n + 1$. Then,

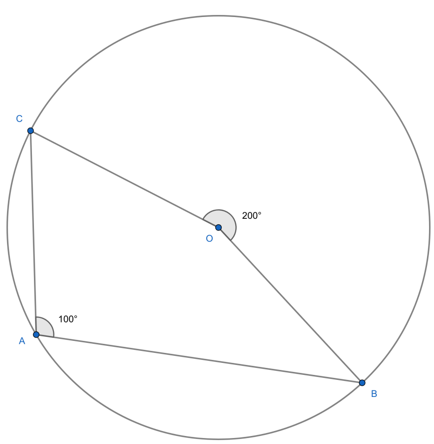
$$\left(1 + \frac{1}{n+1}\right)^{n+1} < \left(1 + \frac{1}{n}\right)^{n+1} < n\left(1 + \frac{1}{n}\right) = n + 1.$$

The correct answer is D).

4) Let K, L, M, and N denote vertices of the newly created quadrilateral. It is easy to see that KLMN is a rectangle - the whole construction is symmetric about the diagonals of the original rhombus, and the diagonals are perpendicular (see Figure). To prove that the rectangle is a square, consider the quadrilateral AKBO. This quadrilateral consists of two right biangles, and so it can be inscribed in a circle whose diameter is AB. Angle KOB is measured by Arc (K,B), and so we have : measure(Angle KOB) — 45 degrees. This shows that the diagonal of KLMN makes a 45 degree - angle with the perpendicular side-bisector of the rectangle KLMN, and so KLMN has to be a square. The correct answer is D).

5) Let us state two theorems that will be handy in our proof:

Theorem 1 (Central Angle Theorem) *The measure of the inscribed angle $\angle BAC$ is half the measure of the central angle $\angle BOC$*

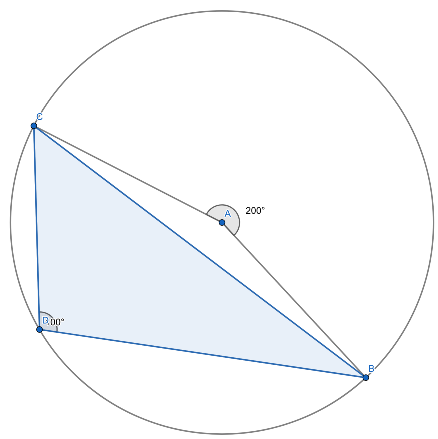


Theorem 2 (Interior Angle Theorem) *The sume of the measures of the interior angles of a triangle is equal ot* $180°$

Remember that triangles can be classified in terms of its sides and in terms of its angles, for this proof we will have to use the classification by its angles: *Obtuse, Right*, and *Acute*.

Case Obtuse

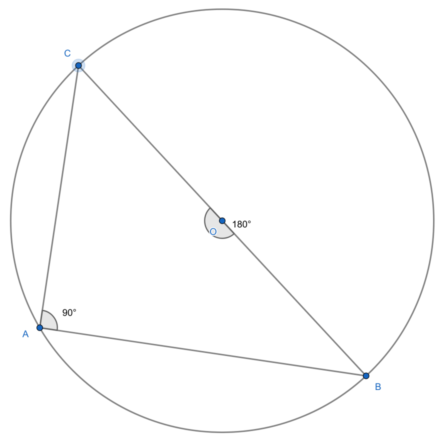
If the triangle $\triangle BCD$ is obtuse, then one of its angles must be greater than $90°$. Let us assume that such angle is $\angle BAC$ (see figure below). Since $m\angle BAC > 90°$, by the Central Angle Theorem $m\angle BOC > 180°$.



Therefore O must be outside the interior of $\triangle BAC$.

Case Right

If the triangle $\triangle BCD$ is right, then one of its angles must be equal to $90°$. Let us assume that such angle is $\angle BAC$ (see figure below). Since $m\angle BAC = 90°$, by the Central Angle Theorem $m\angle BOC = 180°$.



Therefore O must be on the border of $\triangle BAC$ (not on the interior). In none of the previous cases the triangle contains the center of the circunscribing circle, therefore the sample space where we need to work in order to compute the desired probability is the space of acute triangles.

Case Acute

If the triangle $\triangle BCD$ is neither obtuse nor right, then all of its angles must be less than 90°. If we denote by A, B and C the three angles of the triangle, we must have that $m\angle A < 90$, $m\angle B < 90$, and $m\angle C < 90$. The interior angle theorem states that $m\angle A + m\angle B + m\angle C = 180°$. The figure below is a representation of the sample space (in blue) and the inside it is the space of all acute triangles. It is easy to see that the desired probability is $\frac{1}{4}$. The correct answer is A).

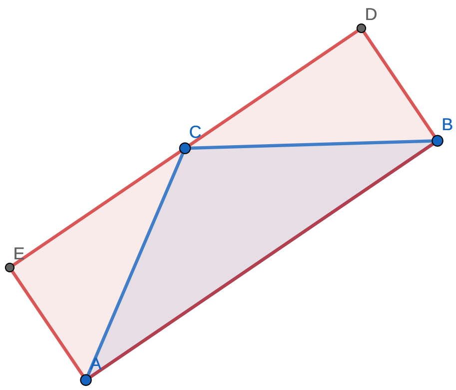
6) The smallest a, i.e., a=1, gives the largest x=4224. The smallest x that gives the natural number a is x=3. The correct answer is B).

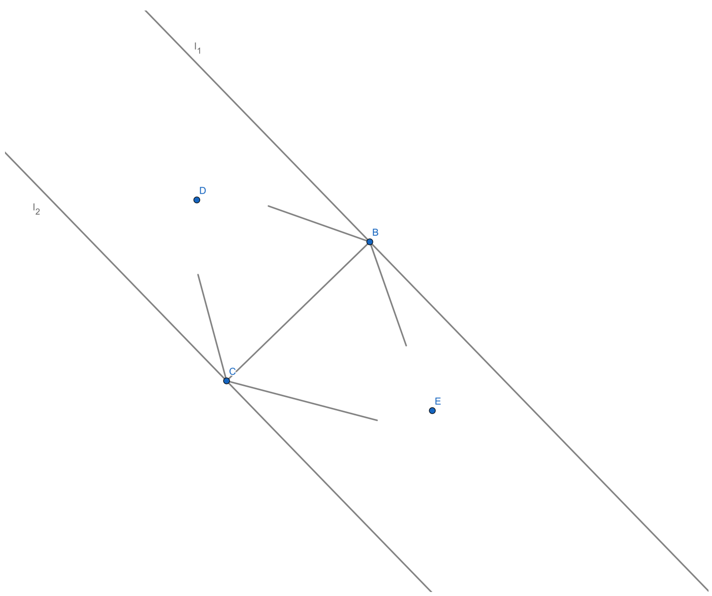
7) The correct answer is E).

8) 4+5+...+n=2010, n(n-1)/2-6=2010, n=60. The correct answer is D).

9) We will show something a little bit stronger, that there exists a rectangle of area 2 containing the polygon. **Case** $n = 3$ In this case we are dealing with a triangle. Without loss of generality suppose that AB is the longest side. The projection of the third vertex C lies in the segment AB, therefore the result is obvious since the rectangle with base AB and parallel side passing through C has area equal to 2 and certainly contains the given triangle (see figure).

Case $n > 3$ If the polygon has more than three vertices, then choose the two vertices of the polygon which are the furthest apart. Call them B and C. Draw perpendiculars to the line BC at B and C, call these lines l_1 and l_2 respectively. Observe that all the vertices should lie between these two lines (otherwise there would be two vertices further apart than $|BC|$).



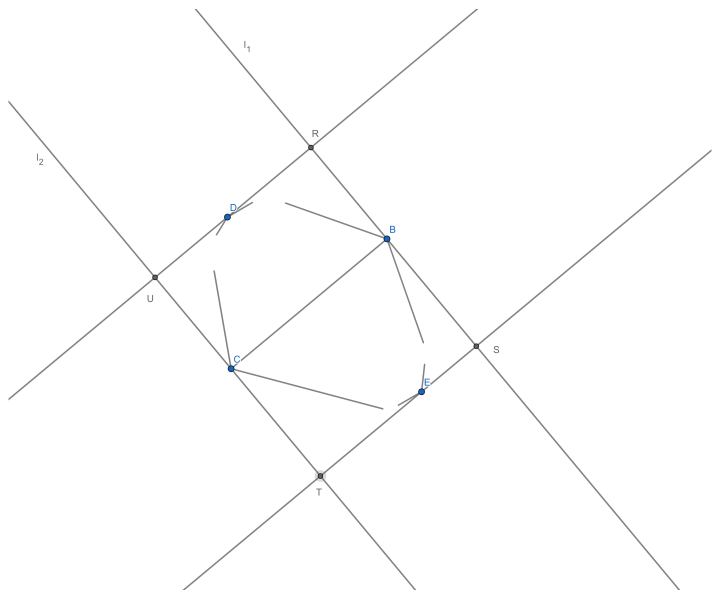


Now consider the smallest rectangle which circumscribes the polygonb and with one pair of ooposites sides lying on l_1 and l_2. Suppose this rectangle touches the polygon again at D and E.

Let the vertices of the rectangle be R, S, T, and U with D on UR and E on ST. Then it is easy to see that area$(RUCB) = 2$area$(\triangle BCD$ and area$(BSTC) = 2$area$(\triangle BEC$ since

$BC||RU$ and $BC||ST$. Finally we have

$$\begin{aligned} area(RSUT) &= 2area(CDBE) \\ &\leq 2\text{area of polygon} \\ &= 2 \end{aligned}$$



10) $(2n-3)(2n-1)(2n+1)(2n+3)+16 = (4n^2-1)(4n^2-9)+16 = 16n^4-40n^2+25 = (4n^2-5)^2$ for all $n \geq 2$. (4p for the first expression, 3p for the third, and 3p for the fourth(final) expression)