1^{st} **Annual Mathematics Olympiad**

Individual Competition

March 4, 2016

Name: ________________________

Justify all answers!

1. Solve

$$|5x - |4x + |3x - |2x + |x||||| = 2016$$

2. Let $8x + 3y = 2016$, and let (x_1, y_1), (x_2, y_2), $\cdots$, (x_n, y_n) be solution pairs that are Natural numbers. Find the sum $x_1 + x_2 + \cdots + x_n$.

3. Find all Natural numbers whose square equals cube of the sums of its digits.

4. Solve $3^n - 2^m = 1$, where n and m belong to the set of Natural numbers.

5. Let $ABCD$ be a square and let M, N, P, and Q be points on the sides AB, BC, CD, DA respectively. If $AM = NC = PD = QA$, prove that $\angle PNC = \angle NQM$.

Solutions:

1. $x = 2016, -\frac{2016}{5}$ (10 points each solution)

2. Since $8x = 2016 - 3y = 3(672 - y)$, then $x = 3k$ (**5 points**). Now $24k = 3(672 - y)$, or $y = 672 - 8k$ (**5 points**). Since x_n, $y_n > 0$, we have $1 \leq k \leq 83$ (**5 points**). Therefore, $\sum_{i=1}^{83} x_i = \frac{3 \cdot 83 \cdot 84}{2} = 10458$ (**5 points**).

3. $n = 27$. $n^2 = m^3 = m^2 \cdot m$, and $m \leq 18$ if m is two-digit number, $m \leq 27$ if m is three-digit number. Therefore, $m = p^2$ for some integer p (**8 points**), or more precisely $m = 4, 9, 16, ...$(**8 points**). Trying out each one of these it is easy to see that $m = 9$ is the only choice, hence $n = 27$ (**4 points**).

4. Answer: $(1, 1)$, $(2, 3)$

$$2^m = 3^n - 1 = (3 - 1)(3^{n-1} + 3^{n-2} + \cdots + 3 + 1)$$

So either $n = 1$ (**8 points**) or $n = 2p$. In the second case we get

$$2^m = 9^p - 1 = (9 - 1)(9^{p-1} + 9^{p-2} + \cdots + 9 + 1).$$

Again, either $p = 1$ (**8 points**) or $p = 2q$, but the second case is impossible since 80 does not divide 2^m (**4 points**).

5. Triangle CNP and DPQ are congruent. To see this notice that they are both right triangles and $CN = DP$, $CP = DQ$, so $NP = PQ$ and these two sides make a right angle. Therefore, $\angle QNP = 45° = \angle AQM$ (**8 points**). $\angle CNQ$ and $\angle NQA$ are equal because they are on transversal (**8 points**), so $\angle CNP = \angle CNQ - 45° = \angle NQA - 45° = \angle NQM$ (**4 points**).