2^{nd} **Annual Mathematics Olympiad**

Group Competition

Barry University

March 31, 2017

School Name: ____

Find all natural numbers a and b such that the following are satisfied:

$$a \mid b^2$$
$$b \mid a^2$$
$$a + 1 \mid b^2 + 1.$$

Please justify your answer!

Solution:

All pairs of the form $(a,b)=(t^2,t)$, (t^2,t^3), $(t^2(t^2-1),t(t^2-1)^2)$, where $t\in\mathbb{N}$.

Let $b^2=ca$. Then $b^2=ca\,|\,a^4$ and $a+1\,|\,ca+1$, which is equivalent to

$$c\,|\,a^3 \quad and \quad a+1\,|\,c-1.$$

Let $c=d(a+1)+1$, $d\in\mathbb{N}_0$. Since $a^3\equiv -1 \pmod{(a+1)}$, we have $\frac{a^3}{c}\equiv -1 \pmod{(a+1)}$, i.e., $\frac{a^3}{c}=e(a+1)-1$ for some $e\in\mathbb{N}$. Hence, $a^3=(d(a+1)+1)(e(a+1)-1)$ which after multiplying and diving by $(a+1)$, becomes $a^2-a+1=de(a+1)+(e-d)$. Then we get $e-d\equiv a^2-a+1\equiv 3 \pmod{(a+1)}$, therefore

$$e-d=k(a+1)+3 \quad and \quad de=a-2-k \ \ (k\in\mathbb{Z}). \tag{1}$$

Consider the following cases:

(1) $k\notin\{-1,0\}$. In this case, we have $de<|e-d|-1$ which is possible if $d=0$. Then $c=1$ and $b^2=a$, so $(a,b)=(t^2,t)$.

(2) $k=-1$. Then $a=d+1$, so $c=a^2$ and $b^2=a^3$. Hence, $(a,b)=(t^2,t^3)$.

(3) $k=0$. Then from (1) we get $a=d^2+3d+2$. Now $c=d(a+1)+1=(d+1)^3$ and $b^2=ca=(d+1)^4(d+2)$. Therefore $d+2=t^2$ for some $t\in\mathbb{N}$ which gives $(a,b)=(t^2(t^2-1),t(t^2-1)^2), t\geq 2$.