2^{nd} Annual Mathematics Olympiad
Barry University

Individual Competition

March 31, 2017

Name: ________________________________

School: ________________________________

Justify all answers!

1. In this sentence, the number of occurrences
of the digit 0 is _____,
of the digit 1 is _____,
of the digit 2 is _____,
of the digit 3 is _____,
of the digit 4 is _____,
of the digit 5 is _____,
of the digit 6 is _____,
of the digit 7 is _____,
of the digit 8 is _____, and
of the digit 9 is _____.

Fill in the blank with numbers so that the sentence is true.

2. At a basketball tournament there was 10 teams. Every team played with every other team exactly once. If at the end of the tournament, team one had x_1 wins and y_1 losses, team two had x_2 wins and y_2 losses, etc. Prove that

$$x_1^2 + x_2^2 + \cdots + x_{10}^2 = y_1^2 + y_2^2 + \cdots + y_{10}^2.$$

3. Three statements are given:

(1) Number $n + 29$ is a perfect square of a Natural number;

(2) Last digit of number n is 8;

(3) Number $n - 60$ is a perfect square of a Natural number.

Find Natural number n if it is known that two of the three statements are correct and one is incorrect.

4. Let $ABCD$ be a square and let M, N, P, and Q be points on the sides AB, BC, CD, DA respectively. If $AM = NC = PD = QA$, prove that $\angle PNC = \angle NQM$.

5. What is the greatest number of bishops which can be arranged on an ordinary chessboard (8x8) in such a way that none of them controls the square on which another lies? (the Bishop can move diagonally, all the way until it meets an obstacle - end of board or a piece).

Solutions:

1. First way: In this sentence, the number of occurrences
of 0 is 1,
of 1 is 7,
of 2 is 3,
of 3 is 2,
of 4 is 1,
of 5 is 1,
of 6 is 1,
of 7 is 2,
of 8 is 1, and
of 9 is 1.

Second way: In this sentence, the number of occurrences
of 0 is 1,
of 1 is 11,
of 2 is 2,
of 3 is 1,
of 4 is 1,
of 5 is 1,
of 6 is 1,
of 7 is 1,
of 8 is 1, and
of 9 is 1.
(**20 points for either solution**)

2. Note that all teams played 9 games, hence $x_i + y_i = 9$, for all $1 \leq i \leq 10$. Since in every match there is exactly one winner, we have that $x_1 + \cdots + x_{10} = \binom{10}{2} = 45$ (**5 points**) which is the total number of matches. Then,

$$\begin{aligned} y_1^2 + y_2^2 + \cdots + y_{10}^2 &= (9 - x_1)^2 + (9 - x_1)^2 + \cdots + (9 - x_{10})^2 \\ &= 10 \cdot 81 - 18(x_1 + \cdots + x_{10}) + x_1^2 + x_2^2 + \cdots + x_{10}^2 \\ &= x_1^2 + x_2^2 + \cdots + x_{10}^2 \qquad \textbf{15 points} \end{aligned}$$

3. Assume that statements 2) and 3) are true. Hence, there exist Natural numbers p and a, such that

$$n - 60 = p^2 \quad \text{and} \quad n = \overline{a8}$$

Then $n - 60$ has last digit 8 and it's a perfect square. A contradiction! Square of a natural numbers ends in 0, 4, or 6.
Next, assume that statements 1) and 2) are true, i.e.,

$$n + 29 = q^2 \quad \text{and} \quad n = \overline{a8}$$

for some Natural numbers p and a. Then $\overline{a8} + 29$ has last digit 7 and it's a perfect square. A contradiction! Last digit of a odd number can end in 1, 5, or 9. Based on the last two contradictions we know that the statement number 2) is incorrect (**12 points**).

Therefore, $n - 60 = p^2$ and $n + 29 = q^2$, for some natural numbers p and q. Subtracting the two equations we get

$$q^2 - p^2 = (q - p)(q + p) = 89 = 1 \cdot 89$$

Hence, $q - p = 1$ and $q + p = 89$. Solving the systems we get p=44 and q=45. Then, $n = 1996$. (**8 points**)

4. Note that $\triangle$ CNP $\cong$ $\triangle$ DPQ (they are both right triangles, CN=DP, and CP=DQ), so NP=PQ and $\angle NPQ = 90°$. Therefore, $\angle QNP = 45° = \angle AQM$ **(8 points)**. $\angle CNQ$ and $\angle NQA$ are on transversal and equal **(8 points)**, so $\angle CNP = \angle CNQ - 45° = \angle NQA - 45° = \angle NQM$ **(4 points)**.

5. 14 (**20 points**)